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Random Walks in Probability Theory

Programme Leader:

Dr. A.P.N. Weerasinghe
Visiting Scientist, Institute of Fundamental Studies and
Assistant Professor, Dept. of Mathematics, Iowa State University, U.S.A

RANDOM WALKS IN PROBABILITY THEORY

In this section we intend to learn about certain probabilistic models and related mathematical computations.

Example 1:

Let us consider the following simple random experiment. A player tosses a coin. Then either 'Heads' or 'Tails' side will appear.

Say, probability [Heads] = p and hence the probability [Tails] = (1 - p) where $0<p<1$; For example, if $p=\frac{1}{2}$ we say the coin is a fair coin. Now let's consider the following game.

The player receives one rupee if the outcome is 'Heads' (H) and the player has to pay one rupee if it is 'Tails' (T). So this can be formulated in the following way.

Let X be the amount player is going to win or to loose

Therefore $X = +1$ if the outcome is 'H'

$= -1$ if the outcome is 'T'

This means $P[X=+1] = p$ and $P[X=-1] = (1-p)$. Certainly the value of X depends on this random experiment. One natural question is what is the average value or the expected value of this X? In other words, how much will the player gain or loose after one game? If p is very near to 1 say $p = 0.9$, it is very likely that this average value is very close to +1 and similarly if p is very near to 0, say $p=0.01$, then it is very probable that the average value of X is very close to -1.

We will denote this average value of X by E[X] and define E[X] by

$$\begin{aligned} E[X] &= 1.\text{prob}\,[X=+1] + (-1)\,\text{prob}\,[X=-1] \\ &= 1.p + (-1)\,(1-p) \\ &= (2p-1) \end{aligned}$$

This E[X] also known as the expected value of X.

Notice that if $p = \frac{1}{2}$ then $E[X] = 0$

$p > \frac{1}{2}$ then $E[X] > 0$ and if $p < \frac{1}{2}$ then $E[X] < 0$.

Example 2:

Let's compute the average value for another random quantity in a random experiment.

Consider a die which has 6 faces. The six faces were numbered from 1 to 6, If you toss this die one of these numbers will be on the top face.

First let's consider a fair die. That means all the faces are equally probable. The player wins or looses money according to the following table.

Number on the top face	1	2	3	4	5	6
X=player's fortune in rupees	5	-4	-1	2	-4	2

How much will the player gain or loose in average after one game?

Let's compute the probabilities first; X takes the values -4, -1, 2 or 5.

$$\text{Prob } [X = 5] = \text{Prob } [\text{Face } 1] = \tfrac{1}{6}$$

$$\text{But, } \Pr[X = -4] = \text{Prob } [\text{Face 2 or Face 5}]$$
$$= \text{Prob } [\text{Face } 2] + \text{Prob } [\text{Face } 5]$$
$$= \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{3}$$

$$\text{Prob } [X = -1] = \text{Prob } [\text{Face } 3] = \tfrac{1}{6}$$

$$\text{Prob } [X = 2] = \text{Prob } [\text{Face 4 or Face 6}]$$
$$= \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{3}$$

Now we are in a position to compute the expected value or the average value of X.

$$\text{So, } E[X] = (5)\, \tfrac{1}{6} + (-4).\, \tfrac{1}{3} + (-1).\, \tfrac{1}{6} + (2)\, \tfrac{1}{3}$$
$$= \tfrac{5}{6} - \tfrac{4}{3} - \tfrac{1}{6} + \tfrac{2}{3}$$
$$= 0.$$

$E(X) = 0$ means this is a fair game for the player. If $E(X) > 0$ then the game is favourable for the player and if $E(X) < 0$ then the game is unfavourable for the player.

Such a quantity X whose value depends on the outcome of a random experiment is called a random variable.

Notice that the average value of a sum of two random variables is equal to the sum of the average values. This means if X and Y are two random variables then $E(X+Y) = E(X) + E(Y)$.

Example;

Let's toss a fair die and say there are two players their fortunes after the game were represented by the random variables X and Y

Number on the top face	1	2	3	4	5	6
X=Fortune for the player 1	5	-8	-2	1	-1	6
Y=Fortune for the player 2	-4	1	-2	2	-4	4
Total amount X + Y	1	-7	-4	3	-5	10

$$E(X+Y) = 1.\tfrac{1}{6} + (-7).\tfrac{1}{6} + (-4).(\tfrac{1}{6}) + (3).\tfrac{1}{6} + (-5).\tfrac{1}{6} + 10.(\tfrac{1}{6}) = -\tfrac{2}{6} = -\tfrac{1}{3}$$

Further $$E(X) = (5)\tfrac{1}{6} + (-8)\tfrac{1}{6} + (-2)\tfrac{1}{6} + (1)\tfrac{1}{6} + (-1).\tfrac{}{6} + (6)\tfrac{1}{6} = \tfrac{1}{6}$$

$$E(Y) = (-4)\tfrac{1}{6} + (1)\tfrac{1}{6} + (-2)\tfrac{1}{6} + (2)\tfrac{1}{6} + (-4)\tfrac{1}{6} + (4)\tfrac{1}{6} = -\tfrac{3}{6} = -\tfrac{1}{2}$$

Clearly, we can see that $E(X+Y) = E(X)+E(Y)$ for this example.

Concept of independance

Consider the following random experiment of tossing a fair coin and tossing a fair die at once. Coin-toss will give one of the two outcomes of Heads (H) or Tails (T). Tossing the die will give one of the six numbers from 1, 2, 3, 4, 5 or 6. Let A be the event of getting H from the coin toss. Let B be the event of getting 5 from tossing the fair die.

We would like to compute the following probabilities

1. Prob [A], 2. Prob [B], 3. Prob [A and B]

Clearly Prob $[A] = \frac{1}{2}$ Since the outcome from tossing the die has no effect on coin-tossing.

Similarly Prob $[B] = \frac{1}{6}$ As coin-toss and tossing the die are independent.

To compute the Prob (A and B) we have to look at all possible outcomes. The event A and B means getting H from the coin and getting 5 from the die. For simplicity, we can write this favourable outcome as (H,5).

All possible outcomes can be written in this form

(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)

Notice each of these outcomes are of equal probability

For eg. Prob of getting (H,2) $= \frac{1}{12}$ (One favourable outcome out of 12 possible outcome)

Similarly Prob [getting (H,5)] $= \frac{1}{12}$

This means Prob [A and B] $= \frac{1}{12}$

= Prob [A] Prob [B]

If A and B are two random events and Prob [A and B] = Prob [A] Prob [B] then we say A and B are independent of each other. Quite often, it is intuitively easy to understand the independent events. Suppose that you tossed a coin two times. Then the two outcomes are independent since the first toss has no effect on the second toss. In other words, the knowledge you gained from the first toss does not increase or decrease your chances of getting Heads from the second toss.

Coin – tossing and difference equations

Example:

Consider the following simple game. The player A tosses a fair coin. So

Prob (Heads) = Prob (Tails) = $\frac{1}{2}$

Before each toss, the player B tries to guess the outcome Heads or Tails. What is the probability that the player B has a correct guess at least once in n tosses?

Let P_n = probability that the player B has a correct guess at least once in n tosses.

Then,

P_{n+1} = Prob [Correct guess once in first n tosses]

+ Prob[no corect guess in first n tosses but $(n+1)^{th}$ guess is correct]

So,

$P_{n+1} = P_n$ + Prob [No correct guess in first n tosses].

Prob [$(n+1)^{th}$ guess is correct]

$$P_{n+1} = P_n + (1 - P_n).\ \frac{1}{2}$$

Hence $P_{n+1} = \frac{1}{2} P_n + \frac{1}{2}$

Now we know how to solve this kind of an equation and the answer is of the form;

$$P_n = A \left(\frac{1}{2}\right)^n + 1$$

with $\quad P_1 = \frac{1}{2} \quad$ therefore $A = -1$

$$P_n = \left(1 - \left(\tfrac{1}{2}\right)^n\right)$$

So notice the probability that the player B has no correct guess in n trials is equal to $\left(\frac{1}{2}\right)^n$.

Consider the following game which involves coin-tossing. For this coin, let us assume P[Heads] = p and P[Tails] = 1-p $0<p<1$. Suppose that an individual with initial wealth of k rupees plays the game by tossing this coin. Each time when the player tosses the coin, player's wealth will increase by one rupee if the toss is 'Heads' and it will decrease by one rupee if the toss is 'Tails'.

If the player looses all the money and ends up with zero rupees then, he or she has to quit the game and game is a 'loss' for the player.

Also let's assume player has a goal say, winning N rupees from this game. So if the player wins N rupees then again player will quit the game and this time player has won the game.

We would like to set up the mathematical model and to compute the winning probability for this game.

Let S_n = wealth of the player at time n; (after the n^{th} toss) since the initial wealth is k rupees, we can write $S_o = k$.

$$S_{n+1} - S_n = \begin{cases} +1 \text{ if the } (n+1)^{th} \text{ toss is Heads;} \\ -1 \text{ if the } (n+1)^{th} \text{ toss is Tails;} \end{cases}$$

To represent each toss we shall introduce the random variables $X_1, X_2, \ldots\ldots X_n$,

$$X_n = \begin{cases} +1 \text{ if the } n^{th} \text{ toss is 'Heads';} \\ -1 \text{ if the } n^{th} \text{ toss is 'Tails'} \end{cases}$$

Clearly $S_{n+1} - S_n = X_{n+1}$;

In particular

$S_n = k + X_1 + \ldots\ldots + X_n$;

Notice that these coin-tosses are independent from each other and hence these random variables $X_1, X_2, \ldots X_n$ are independent. This model is called The random walk.

For each n, Prob $[X_n = +1] = p$

and Prob $[X_n = -1] = 1 - p$;

Notice $0 \leq k \leq N$;

We are interested in computing the winning probability of reaching N rupees before loosing all the money and starting from $S_o = k$;

Define;

U_k = Prob [reaching N before 0 given that initial wealth $S_o = k$]

Clearly $U_o = 0$ and $U_N = 1$;

For 0<k<N;

U_k = Prob [winning the game and $S_1 = k+1$, $S_o=k$]

+ Prob [winning the game and $S_1 = k-1$, $S_o=k$]

= p. Prob [winning the game starting from k+1]

+ (1 - p) Prob [winning the game starting from k-1]

Hence

$$U_k = p\, U_{k+1} + (1 - p)U_{k-1}$$

$$U_0 = 0,\ U_N = 1,$$

This is a second order linear difference equation, and it is not too difficult to solve it.

Let's write $q = (1-p)$; then $q.(U_k - U_{k-1}) = p.(U_{k+1} - U_k)$

Now let's label these increments by $V_k = U_k - U_{k-1}$ for $1 \leq k \leq N$

So,

$$q.\ V_k = p.\ V_{k+1} \text{ for } 1 \leq k \leq N$$

$$V_{k+1} = \left(\frac{q}{p}\right)V_k; \quad V_1 = U_1 \ ;$$

Therefore,

$$V_2 = \left(\frac{q}{p}\right)V_1$$

$$V_3 = \left(\frac{q}{p}\right)V_2$$

$$V_r = \left(\frac{q}{p}\right)V_{r-1} \quad \text{for } 1 \leq r \leq N$$

Multiplying them together we get $V_2.V_3...V_r = \left(\frac{q}{p}\right)^{r-1}V_1.V_2...V_{r-1}$

This means $V_r = \left(\frac{q}{p}\right)^{r-1}V_1$ for each r;

Now $U_r = (U_r - U_{r-1}) + (U_{r-1} - U_{r-2}) + ... + (U_1 - U_0) + U_0$

$$= V_r + V_{r-1} + ... + V_1 + U_0$$

Since $U_0 = 0$, $U_r = V_1 + V_2 + ... + V_r$

$$= V_1 + \left(\frac{q}{p}\right)V_1 + ... + \left(\frac{q}{p}\right)^{r-1}V_1$$

$$= [1 + \left(\frac{q}{p}\right) + ... + \left(\frac{q}{p}\right)^{r-1}]V_1$$

$$U_r = \frac{[\left(\frac{q}{p}\right)^r - 1]}{[\left(\frac{q}{p}\right) - 1]}V_1 \quad \text{if } p \neq q$$

$$= rV_1 \quad \text{if } p = q;$$

Recall $V_1 = U_1$;

Now once we derive U_1, we are able to find U_r for all r: $0 \leq r \leq N$;

Since $U_N = 1$ we can use this to find the value of U_1;

$$1 = U_N = \frac{[\left(\frac{q}{p}\right)^n - 1]}{[\left(\frac{q}{p}\right) - 1]}U_1 \text{ if } p \neq q$$

and

$$1 = U_N = N.U_1 \text{ if } p = q \ ;$$

Then, $U_1 = \frac{[\left(\frac{q}{p}\right) - 1]}{[\left(\frac{q}{p}\right)^n - 1]}$ if $p \neq q$.

and

$$U_1 = \frac{1}{N} \text{ if } p = q.$$

Therefore for any r : $0 \leq r \leq N$,

$$U_r = \frac{[\left(\frac{q}{p}\right)^r - 1]}{[\left(\frac{q}{p}\right)^n - 1]} \text{ if } p \neq q$$

$$= \frac{r}{N} \text{ if } p = q$$

This formula gives us the winning probabilities. In the case of a fair coin (i.e. $p = q = \frac{1}{2}$) we see that the winning probability with starting at k is given by $\frac{k}{N}$

Let's ask the following question now.

Consider a player with initial wealth of k rupees. After a random time (which may depend on his position k). The player will win or loose the game. Let's call this "the playing time." It would be interesting to know the average playing time for a player with initial wealth k.

Define T_k = playing time for a player with initial wealth k.

Take V_k = average playing time with initial wealth k $\equiv E[T_k]$

For $k = 0$, $T_0 = 0$ Hence $V_0 = 0$;

$k = N$, $T_N = 0$ So $V_N = 0$;

Exercise: Try to set up the difference equation for V_k.

$$V_k = p(1+V_{k+1}) + q\,(1 + V_{k+1}) \quad \text{where } q = (1 - p)$$

$$V_k = 1 + p\,V_{k+1} + q\,V_{k+1}$$

Can you solve this? Use the same method.

$$q\,(V_k - V_{k+1}) = 1 + p(V_{k+1} - V_k)$$

introduce $R_k = (V_k - V_{k-1})$ for $1 \leq k$;

So, $q\,R_k = 1 + pR_{k+1}$

Now one can solve for R_k. This is left as an exercise.