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Irrational Numbers

Programme Leader:
Dr. U. Mampitiya
Senior Lecturer, University of Kelaniya

IRRATIONAL NUMBERS

0. Introduction

The simplest numbers are the positive whole numbers, 1,2,3, and so on, used for counting. These are called natural numbers.

If any two natural numbers are added, the result will be a natural number. Similarly, if any two are multiplied, the product will be a natural number. However, not every subtraction of one natural number from another gives a natural number. In order to achieve these three properties in a single set of numbers we extend the natural number set by including zero and the negatives. i.e.,

0, -1, -2, -3, -4,.........

The natural numbers together with zero and the negatives form the integers or whole numbers.

The basic necessities of everyday life led to the introduction of common fractions like 1/2, 1/3, 2/3, 5/4, etc. Such numbers are called rational numbers, because they are ratios of whole numbers. When we talk about rational numbers we mean integers and common fractions. Therefore the class of rational numbers contains the class of integers.

The discovery that common fractions are not sufficient for the purpose of geometry was made by the Greeks more than 2500 years ago. They noticed that the length of the diagonal of a square whose sides are one unit long cannot be expressed by any rational number. Today we express this fact by saying that the square root of 2 (which, according to the Pythagorean Theorem, is the length of the diagonal of such a square) is an irrational number. Other irrational numbers occur in various natural ways in elementary mathematics. For example, the circumference of a circle is an irrational multiple, namely .... of the diameter. The value of Sinx when x has the value 60° is the irrational number $\sqrt{3}/2$.

In general a number which is not equivalent to a fraction is said to be irrational.

The real numbers consist of all rational and irrational numbers, and form the central number system of mathematics.

The idea of this note is to present some methods for identifying irrational numbers.

1.Any "terminating" decimal (such as 2.8 or 0.364) is equivalent to a fraction. For example,

2.8 = 28/10 = 14/5
0.364 = 364/100 0 = 91/250
1.414 = 1414/1000 = 707/500

However, not every fraction is equivalent to a terminating decimal.

Example 1. 1/3 = 0.333..., where the three dots indicate an endless sequence of threes. Sometimes one writes $1/3 = 0.\bar{3}$, where the bar over the 3 means that the 3s are repeated indefinitely.

Example 2. 15/11 = 1.363636... = $1.\overline{36}$, Either notation indicates endless repetition of the pair of digits, 36.

A repeating decimal is always equivalent to a fraction.

Example 3. Convert the repeating decimal $0.\overline{135}$ to an equivalent fraction.

Solution: Let x = 0.135135135... .

Then 1000x-x = 135.135135135... - 0.135135135...

= 135.

Thus, x = 135/999 = 15/111 = 5/37.

[Why did we multiply x by 1000?]

Example 4. Convert $3.1\overline{2}$ to an equivalent fraction.

Solution:Let x = 3.12222.....
Then 10x-x = 31.22222... - 3.1222.....
This implies 9x = 28.1.
Thus, x = 28.1/9 = 281/90.

The numbers with non-repeating decimals are irrational.

Example 5. The numbers 1.01001000100001... and 9.3433344333444... are irrational since they are non-repeating decimals. (They cannot be equivalent to fractions).

Problems:

1. Convert each of the following to an equivalent decimal
(a) 3/8 (b) 1/9 (c) 13/11 (d) 4/7 (e) 45/37

2. Convert each of the following to an equivalent fraction in lowest terms
(a) $0.\overline{81}$ (b) $1.\overline{1}$ (c) $24.\overline{109}$ (d) $0.\overline{837}$ (e) $0.\overline{142857}$

3. Describe two irrational numbers other than the ones in Example 5.

4. If a fraction a/b is converted to a decimal what is the maximum possible length of the repeating pattern?

2. We can give indirect proofs to show that some numbers are irrational.

Example 1. Prove that √2 is irrational.

Solution: Suppose that √2 were a rational number, say √2 = a/b, where a and b are integers. We presume that the rational fraction a/b is in its lowest terms. Specifically, we make use of the fact that a and b are not both even. Squaring the above equation, and simplifying we get

$$2 = a^2/b^2, a^2 = 2b^2$$

The term $2b^2$ represents an even integer, so a^2 is an even integer and hence a is an even integer, say a = 2c, where c is also an integer. Replacing a by 2c in the equation $a^2 = 2b^2$, we obtain

$$(2c)^2 = 2b^2, 4c^2 = 2b^2, 2c^2 = b^2$$

The term $2c^2$ represents an even integer, so b^2 is an even integer, and hence b is an even integer. But now we have concluded that both a and b are even integers, whereas a/b was presumed to be in lowest terms.

This contradiction leads us to conclude that it is not possible to express $\sqrt{2}$ in the rational form a/b, and therefore $\sqrt{2}$ is irrational.

Problems:

1. Prove that $\sqrt{3}$ is irrational.
2. Prove that $\sqrt{6}$ and $\sqrt{2}+\sqrt{3}$ are irrational.
3. Prove that $\sqrt{5}$ is irrational.
4. Prove that $\sqrt{15}$ is irrational.

3. It was discussed in the preceding section that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{6}$ are irrational. What we want to do next is establish the irrationality of all such numbers by a common scheme instead of treating them one at a time.

We recall that a root of an equation in x is a value which, when substituted for x, satisfies the equation. For example $\sqrt{7}$ is a root of $x^2-7 = 0$ since

$$(\sqrt{7})^2-7 = 7-7 = 0.$$

Example 1. Prove that $\sqrt{7}$ is irrational.

Solution: $\sqrt{7}$ is a root of $x^2-7 = 0$. If $\sqrt{7} = a/b$, where a and b are integers with no common factors. Then $(a/b)^2-7 = 0$, $a^2-7b^2 = 0$, $a^2 = 7b^2$.

Now b is a factor of $7b^2$ implies that b is a factor of a^2 and hence a factor of a. However, if b is a factor of a, then a/b is an integer. i.e. $\sqrt{7}$ is an integer. But this is not possible since;

$$4 < 7 < 9 \text{ implies } \sqrt{4} < \sqrt{7} < \sqrt{9} \text{ implies } 2 < \sqrt{7} < 3$$

i.e., $\sqrt{7}$ lies between the consecutive integers 2 and 3. We conclude that $\sqrt{7}$ cannot be written as a fraction and hence it is irrational.

Example 2. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Solution: Let $x = \sqrt{2} + \sqrt{3}$. Then $x-\sqrt{2} = \sqrt{3}$ and squaring both sides we obtain $x^2-2x\sqrt{2} + 2 = 3$ or $x^2-1 = 2x\sqrt{2}$. If we square this again, we obtain $x^4-2x^2+1 = 8x^2$ or $x^4-10x^2+1 = 0$.

Now we know $\sqrt{2} + \sqrt{3}$ is a root of this equation. If $\sqrt{2}+\sqrt{3}$ were a rational then $\sqrt{2}+\sqrt{3} = a/b$ for some integers a and b where a/b is in lowest terms. Then;

$(a/b)^4-10(a/b)^2+1 = 0$ implies $a^4-10b^2a^2 + b^4 = 0$

implies $a^4 = b^2 (10a^2-b^2)$.

Thus b must be a factor of a^4. But than b is a factor of a and hence a/b is an integer. However, we can show (Do!) that $\sqrt{2}+\sqrt{3}$ is not an integer. Hence $\sqrt{2}+\sqrt{3}$ must be irrational.

Problems: As in the preceding examples prove that

1. $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$ are irrational.
2. $3\sqrt{5}$, $5\sqrt{91}$ are irrational.
3. $(4\sqrt{13}-3)/6$ is irrational.
4. $1/3\,(2\sqrt{6} + 7)$ is irrational.

4. In this section we shall show, using the methods of the previous section and certain basic trigonometric identities, that for many angles θ the corresponding values of the trigonometric functions are irrational.

Example 1. Prove that $\text{Cos}20^\circ$ is irrational.

Solution: Note that the formula

$\text{Cos}3\theta = 4\,\text{Cos}^3\theta - 3\,\text{Cos}\theta$ gives us
$\text{Cos}60^\circ = 4\text{Cos}^3 20^\circ - 3\,\text{Cos}20^\circ$.

If we write $x = \text{Cos}\ 20^\circ$, and make use of the fact that $\text{Cos}60^\circ = 1/2$, we get

$1/2 = 4x^3-3x$ or $8x^3-6x-1 = 0$.

Because of the construction we know that $\text{Cos}20^\circ$ is a root of the above equation. Hence if $\text{Cos}20^\circ$ were rational, say,

$\text{Cos}20^\circ = a/b$, then $8(a/b)^3 - 6(a/b)-1 = 0$.

or $8a^3 - 6ab^2 - b^3 = 0$.

implies $8a^3 = b^2(6a+b)$.

Thus b must be a factor of $8a^3$. Since b cannot be a factor of a^3 (why?) we get b must be a factor of 8 i.e. $b = \pm1, \pm2, \pm4, \pm8$.

On the other hand $b^3 = a(8a^2 - 6b^2)$ implies that a is a factor of b^3. Thus $a = \pm1$.

Therefore the only possible rational roots of $8x^3-6x-1 = 0$ are $\pm1, \pm1/2, \pm1/4, \pm1/8$. But none of these eight possibilities is an actual root, as can be seen by substitution. Hence we conclude that $\text{Cos}\ 20^\circ$ is an irrational number.

Example 2. Prove that $\sin 10^\circ$ is irrational.

Solution: The trigonometric identity $\cos 2\theta = 1-2\sin^2\theta$ gives us $\cos 20^\circ = 1-2\sin^2 10^\circ$.

Now suppose that $\sin 10^\circ$ were rational. Then both $\sin^2 10^\circ$ and $1-2\sin^2 10^\circ$ also would be rational. But $\cos 20^\circ$ is irrational, as we have already proved. Hence $\sin 10^\circ$ is irrational.

Problems:

Prove that the following numbers are irrational
(1) $\cos 40^\circ$ (2) $\sin 20^\circ$ (3) $\cos 10^\circ$ (4) $\sin 50^\circ$.

5. In this section, we consider the values of logarithmic functions. All the logarithms discussed in this section will be taken to the base 10. We recall that, given a positive real number y its logarithm to base 10 is defined to be a number k such that $10^k = y$. Thus for any y 0,

$\log y = k$ and $10^k = y$ are equivalent statements.

Example 1. Prove that $\log 2$ is irrational.

Solution: Suppose that $\log 2 = a/b$, where a and b are positive integers. This means that

$$2 = 10^{a/b}$$

or $$2^b = 10^a = 2^a 5^a$$

Hence 5 must be a factor of 2^b. But this is not possible. Hence $\log 2$ is irrational.

Example 2. Prove that $\log 21$ is irrational.

Solution: Suppose $\log 21 = a/b$, where a and b are positive integers.
Then $21 = 10^{a/b}$
or $21^b = 10^a$.

implies that 10 is a factor of 21^b. But this is not possible (why?). Hence $\log 21$ is irrational.

Problems:

1. Prove that $\log(3/2)$ is irrational.
2. Prove that $\log 15$ is irrational.
3. Prove that $\log 5 + \log 3$ is irrational.