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Infinite Sums

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Infinite Sums

If we are given two or three numbers to add that will not be a difficult task. However, in the infinite case we have to face many problems. The very first one is the meaning of an infinite sum.

If we are going to consider infinite sums with positive and negative terms then this will be very complicated. Consequently, in this course we pay attention to an infinite sum of non negative numbers.

It is very important to have a notation for each number depending on its position. Let us denote 1st number by a_1 , 2nd number by a_2 and so on. May be you have already heard in this case that (a_n) is called a sequence.

Now assume that we have a sequence of numbers (a_n) . Let S_n denotes the sum of the first n terms of the sequence.

$$\text{i.e. } S_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^n a_k ;$$

Then we take limit of S_n when n approaches infinity (∞). If there is a finite limit for S_n we state this limit by $\lim S_n$ and write

$$\sum_{n=1}^{\infty} a_n = \lim S_n. \text{ Otherwise, we write } \sum_{n=1}^{\infty} a_n = \infty.$$

In addition, in the case when there is a finite limit we say $\sum_{n=1}^{\infty} a_n$ is a convergent series or $\sum_{n=1}^{\infty} a_n$ converges. In the other case we say $\sum_{n=1}^{\infty} a_n$ is a divergent series or $\sum_{n=1}^{\infty} a_n$ diverges.

Examples;

1. Find $\sum_{n=1}^{\infty} 0$.

Observe that in this series every term is 0. Hence, $S_n = 0$ too. Thus, $\lim S_n = 0$.

Therefore, $\sum_{n=1}^{\infty} 0 = 0$.

2. Find $\sum_{n=1}^{\infty} a_n$ for $a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 0, a_5 = 0, a_6 = 0, \dots$

Then $S_1 = 1, S_2 = 3, S_3 = 7, S_4 = 7, S_5 = 7$.

It is easy to see that $S_n = 7$ for each $n \geq 3$ therefore, $\lim S_n = 7$.

Thus, $\sum_{n=1}^{\infty} a_n = 7$.

3. Find $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$.

Remember from 'O' Level classes that this is called a geometric series.

$$\text{Now } S_n = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^n$$

$$\text{and } \frac{1}{2}S_n = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots + \left(\frac{1}{2}\right)^{n+1}$$

Thus by subtracting the second equation from the first equation we get

$$\frac{1}{2}S_n = \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}$$

$$\text{This gives us, } S_n = 1 - \left(\frac{1}{2}\right)^n$$

$$\text{and } \lim S_n = \lim \left[1 - \left(\frac{1}{2}\right)^n\right] = 1.$$

$$\text{Hence, } \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1.$$

4. Find $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$.

$$\text{As before, } S_n = \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \dots + \left(\frac{1}{3}\right)^n \text{ and } \frac{1}{3}S_n = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \dots + \left(\frac{1}{3}\right)^{n+1}$$

$$\text{Also, } S_n - \frac{1}{3}S_n = \frac{1}{3} - \left(\frac{1}{3}\right)^{n+1}$$

$$\text{i.e. } \frac{2}{3} S_n = \frac{1}{3} - \left(\frac{1}{3}\right)^{n+1}.$$

$$\text{This yields that } S_n = \frac{1}{2} \left(1 - \frac{1}{3^n}\right).$$

By taking $\lim S_n$ we get;

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{2}.$$

5. Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

$$\text{Observe that } \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

$$\text{Hence, } S_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \dots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1} \text{ and } \lim S_n = 1.$$

$$\text{Finally, we have } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

6. Find $\sum_{n=1}^{\infty} 1$. This means to find the infinite sum when every term is equal to 1.

$$\text{Clearly } S_n = 1 + 1 + \dots + 1 \text{ (n times)}$$

$$= n.$$

Consequently, when n tends to infinity S_n does so, too. In other words $\lim S_n$ is not finite.

$$\text{Thus, } \sum_{n=1}^{\infty} 1 = \infty.$$

7. Find $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

This is an example of an infinite sum which is very hard to find. However, in this problem we can show that this is a convergent series by the following method.

$$\text{It is clear that, } \frac{1}{n^2} \leq \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}, \text{ when } n \geq 2.$$

$$\text{Then, } 0 \leq S_n \leq 1 + 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} - \frac{1}{n} = 2 - \frac{1}{n}.$$

Thus, $\lim 0 \leq \lim S_n \leq \lim \left(2 - \frac{1}{n}\right)$.

Therefore, $\lim S_n \leq 2$.

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

It is clear that given $0 \leq a_n \leq b_n$, if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges or if $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges. In fact this can be shown rigorously and is called comparison test for series.

Using the comparison test example 7 follows immediately.

8. Find $\sum_{n=1}^{\infty} \frac{1}{n}$.

Observe that $S_2 = 1 + \frac{1}{2}$,

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2},$$

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3 \cdot 1}{2},$$

$$S_{16} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{8} + \frac{1}{9} + \dots + \frac{1}{16} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{16} + \dots + \frac{1}{16} = 1 + 4 \cdot \frac{1}{2}.$$

Similarly, it can be shown that $S_{2^n} \geq 1 + \frac{n \cdot 1}{2}$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

9. Show that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Since, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we can use the comparison test.

Clearly, $0 \leq \frac{1}{n^3} \leq \frac{1}{n^2}$ and by comparison test $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Exercises:

1) Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.

2) Find $\sum_{n=1}^{\infty} (2/3)^n$.

3) Find $\sum_{n=1}^{\infty} \frac{62}{(100)^n}$. Observe that this is nothing but the repeating decimal .62.

4) Discuss the convergence of $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$.

5) Discuss the convergence of $\sum_{n=1}^{\infty} \frac{1}{(2n+1)}$.

6) If (a_n) is any sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n^2$ converges show that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges, too.

7) If (a_n) is any sequence of positive numbers then, $\sum_{n=1}^{\infty} \frac{a_n}{n^3} + \frac{n}{a_n}$ diverges.

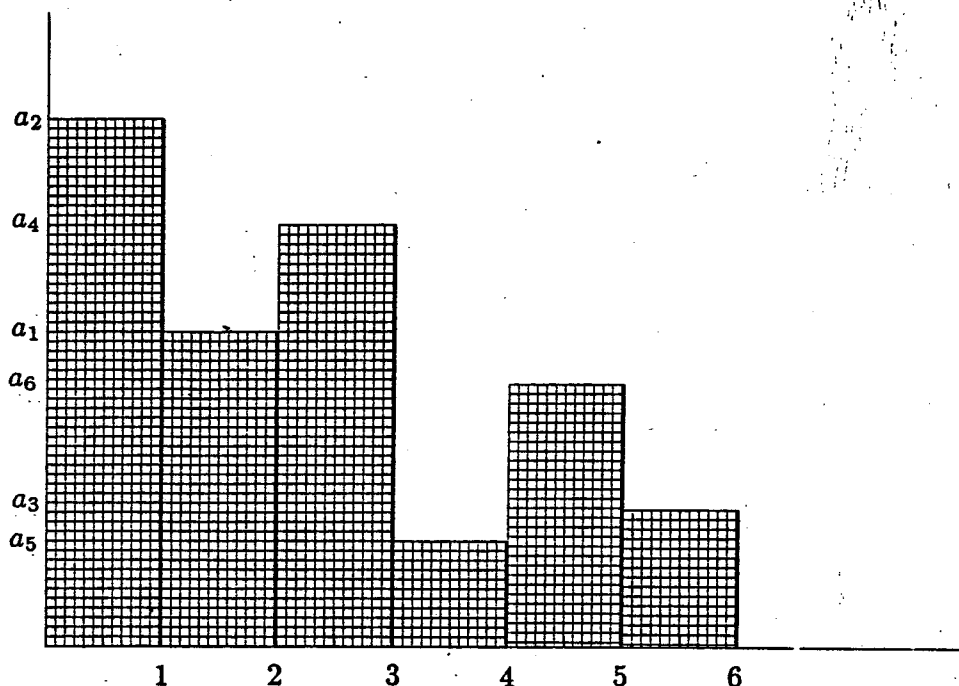
8) If (α_n) is any sequence of positive numbers then $\sum_{n=1}^{\infty} \frac{\alpha_n^2}{n+1} + \frac{n^2}{\alpha_n^3} + \frac{\alpha_n}{n^2(n+1)}$ diverges.

9) Is it necessarily true that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges for a sequence of positive terms?

10) Is it necessarily true that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \sqrt[3]{a_n}$ converges for a sequence of positive terms?

Now we are going to see that an infinite sum of positive terms can be regarded as an area.

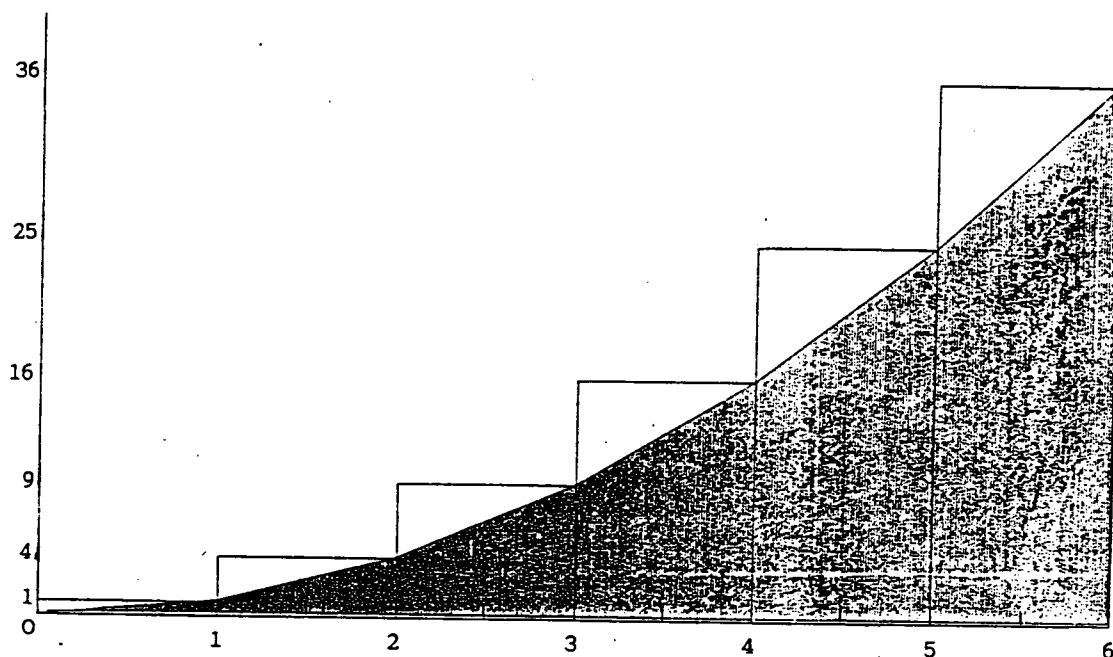
Let us start with the sequence (a_n) of positive terms. Let us graph the step function as follows.



Observe that the area of first rectangle is a_1 , area of second rectangle is a_2 and similarly area of the n^{th} rectangle is a_n .

Hence it is clear that S_n is the sum of the areas of the first n rectangles. This implies that the area between the graph and the X-axis from origin to infinity is $\sum_{n=1}^{\infty} a_n$.

Example 10: Show that the area enclosed by $y = x^2$, x - axis, y - axis and the line $x = 6$ is less than 91 units.



From the graph it is clear that the required (Shaded) area is less than sum of the areas of the 6 rectangles.

Area of the 1st rectangle is 1^2 and area of the second rectangle is 2^2 etc.

Hence, sum of the areas of the 6 rectangles is given by $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = \frac{6(6+1)(2 \cdot 6+1)}{6} = 7.13 = 91$.

Infact this is what we were looking for.

Exercises:

11) I am sure that you have read in the newspaper, Daily news Wednesday July 11, 1990 that the world's richest two men are from Japan. Richest man is Yoshiaki Tsutsumi (16 billion U.S. dollars) and the second richest man is Taikichiro Mori (14.6 billion U.S. dollars). Explain how to construct a garden with area square mile so that even these two men together do not have enough money to put a fence around.

12) Construct a similar garden in a square of one square mile with area less than that of the square.