14 September 1991

Maxima and Minima with no Calculus

Programme Leader:
Dr. W. Ramasinghe
Visiting Research Associate, Institute of Fundamental Studies and
Assistant Professor of Mathematics, Ohio Northern University, U.S.A

MAXIMA AND MINIMA WITH NO CALCULUS

In this course we try to find maxima and minima of certain functions without appealing to calculus.

Definition Let $a_1, a_2, ..., a_n$ be positive numbers. Then $\frac{a_1+a_2+....+a_n}{n}$ is called the arithmetic mean and $\sqrt[n]{a_1a_2....a_n}$ is called the geometric mean of the numbers $a_1, a_2, ..., a_n$.

Theorem: For positive numbers $a_1, a_2, ..., a_n$, $\sqrt[n]{a_1a_2.....a_n} \leq \frac{a_1+a_2+....+a_n}{n}$ and equality holds if and only if $a_1 = a_2 = a_3 = ... = a_n$.

Proof: We will give the proof for n = 2, 3, 4 and general case is left to the reader as an exercise.

case (i): n = 2. It is clear that,

$$\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \geq 0 \text{ and equality holds if and only if } \sqrt{a_1} - \sqrt{a_2} = 0.$$

By simplifying this we get $\frac{a_1+a_2}{2} \geq \sqrt{a_1a_2}$ and equality holds if and only if $a_1 = a_2$.

case (ii): n = 4. $\frac{a_1+a_2+a_3+a_4}{4} = \frac{\frac{a_1+a_2}{2}+\frac{a_3+a_4}{2}}{2} \geq \sqrt{\frac{(a_1+a_2)}{2}.\frac{(a_3+a_4)}{2}}$ by case (i).

Applying case (i) again we get;

$$\frac{a_1+a_2+a_3+a_4}{4} \geq \sqrt{\sqrt{a_1a_2}\sqrt{a_3a_4}} = \sqrt[4]{a_1a_2a_3a_4}.$$

First inequality is an equality if and only if $\frac{a_1+a_2}{2} = \frac{a_3+a_4}{2}$.

Second inequality is an equality if and only if $a_1 = a_2$ *and* $a_3 = a_4$.

Combining all above we get;

$$\frac{a_1+a_2+a_3+a_4}{4} = \sqrt[4]{a_1a_2a_3a_4} \text{ if and only if } a_1 = a_2 = a_3 = a_4.$$

This completes the proof in the case n = 4.

case (iii): n = 3. From case (ii), $\frac{a_1+a_2+a_3+\frac{(a_1+a_2+a_3)}{3}}{4} \geq \sqrt[4]{a_1a_2a_3\frac{(a_1+a_2+a_3)}{3}}$.

i.e. $\frac{a_1+a_2+a_3}{3} \geq \sqrt[4]{a_1a_2a_3}\sqrt[4]{\frac{a_1+a_2+a_3}{3}}$.

i.e. $\left(\sqrt[4]{\frac{a_1+a_2+a_3}{3}}\right)^3 \geq (\sqrt[4]{a_1a_2a_3})$.

Thus $\frac{a_1+a_2+a_3}{3} \geq \sqrt[3]{a_1a_2a_3}.$

Equality holds if and only if $a_1 = a_2 = a_3$ from the very first line. We leave the proof of the general case for the reader. (Hint: First prove the result for $n = 2^m$.)

Examples.

1. Find the maximum value of $x\sqrt{9-x^2}$ with x between 0 and 3.

Solution: Since x is between 0 and 3 both x and $\sqrt{9-x^2}$ are positive numbers.

Hence by case (i) of the proof,

$x\sqrt{9-x^2} = \sqrt{x^2(9-x^2)} \leq \frac{x^2+9-x^2}{2} = \frac{9}{2}$ and equality holds if and only if $x^2 = 9 - x^2$ and hence $x = \frac{3}{\sqrt{2}}$.

Hence, the maximum of $x\sqrt{9-x^2}$ with x between 0 and 3 is $\frac{9}{2}$ and it happens when x is equal to $\frac{3}{\sqrt{2}}$.

2. Find the minimum of $x + \frac{400}{x}$ when x is positive.

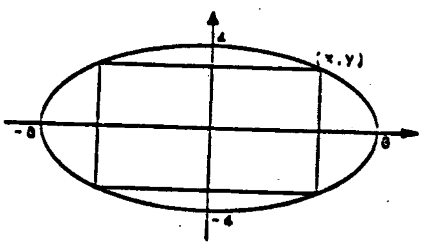
Solution: Since x is positive both x and $\frac{400}{x}$ are positve numbers.

Hence by case (i) of the proof,

$x + \frac{400}{x} \geq 2\sqrt{x.\ \frac{400}{x}} = 40$ and equality holds if and only if $x = \frac{400}{x}$ ($i.e\ \ x = 20$).

Thus minimum of $x + \frac{400}{x}$ with x positive is 40 and it happens when x is 20.

3. Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$, with sides parallel to the axes.



Consider the rectangle inscribed in the ellipse as shown in the picture. Then sides are of 2x and 2y length and the point (x,y) lies on the ellipse. Let A be the area.

Then $A = 2x.2y = 4xy = 4x.\ 4\sqrt{1-\frac{x^2}{9}}$

$= 16.3\sqrt{\frac{x^2}{9}}.\sqrt{1-\frac{x^2}{9}}$
$\leq 48\frac{\frac{x^2}{9}+1-\frac{x^2}{9}}{2} = 24.$

and equality holds if and only if $\frac{x^2}{9} = 1-\frac{x^2}{9}$ if and only if $x = \frac{3}{\sqrt{2}}$ *and* $y = \frac{4}{\sqrt{2}}$.

Hence, the required area is 24 and the point of this rectangle in the ellipe in first quadant is $(\frac{3}{\sqrt{2}}, \frac{4}{\sqrt{2}})$

Exercises:

1) If $a_1, a_2, ..., a_n$ are positive real numbers.
   then,

   i. $\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+.............+\frac{1}{a_n}} \leq \sqrt[n]{a_1a_2...a_n}$

   ii. When does the equality hold in (i)? The quantity in the left hand side of (i) is called the harmonic mean of $a_1, a_2, .., a_n$.

2) Find the volume of the largest rectangular solid that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ edges parallel to the axes.

3) Find the minimum value of $\frac{18}{x}+\frac{12}{y}+xy$ taken over all positve values of x and y.

4) Find the minimum value of $\frac{98}{x}+\frac{24}{y}+\frac{36}{z}+\frac{32}{r}+\frac{6}{s}+xyzrs$, taken over all positive values of x,y,z,r,s.

5) Find the minimum value of $\frac{1}{xy}+\frac{4x^2y^2}{z}+z$ taken over all positive values of x,y,z.

6) Find the shortest distance from the origin to the curve $x^2y = 108$.

7) Prove that among all triangles of a given perimeter, the equilateral triangle has the largest area.