Postprocessing solutions of the Electromagnetic Fields

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Part B, Finite Element Formulation

The Poisson equation is the governing equation of electromagnetic fields. That is

$$-\varepsilon\nabla^2\varphi = \rho$$

and

$$-\mu^{-1}\nabla^2 A = J$$

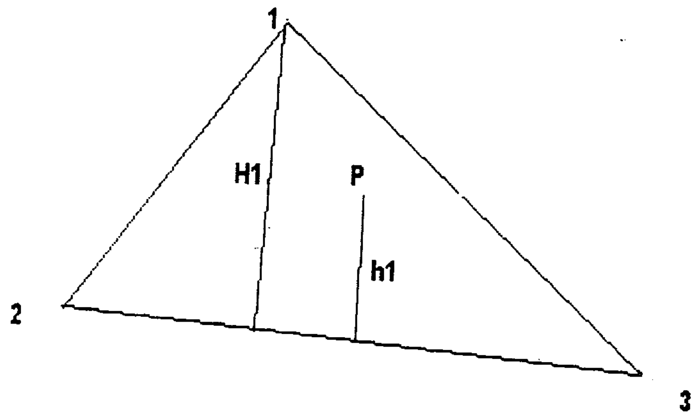
are electric and magnetic fields, where φ electric potential, ε permittivity, ρ electric charge density, μ permeability, A magnetic vector potential and J current density. These equations can be solved the Finite Element Method and we may get postprocess solutions for the solution region (domain). I have used first order triangular finite elements. The two dimensional problem is solved as example.

We can construct a functional, which has a minimum at the point of the solution of the above Poisson equations as

$$L(\varphi) = \iint\{\frac{1}{2}\varepsilon[\nabla\varphi]^2 - \rho\varphi\}dR$$

$$L(A) = \iint\{\frac{1}{2}\mu^{-1}[\nabla A]^2 - JA\}dR$$

It can be easily shown that the minimum of these functional is satisfied by the above Poisson equations. Triangles have been used as basic element shapes. The above functional integrals are the domain is the sum of the sub domains. Now, let see the triangular formulation



Point is P≡ (x, y) = (ξ1, ξ2, ξ3), where ξ1 = h1/H1

The potential within the triangle is determined by first order interpolation.

$$\varphi(x,y) = \xi_1\varphi_1 + \xi_2\varphi_2 + \xi_3\varphi_3 = \alpha\phi$$

Where

$$\alpha = [\xi_1, \xi_2, \xi_3]$$
$$\phi = [\varphi_1, \varphi_2, \varphi_3]$$

In triangle, φ_1 give from vertex 1, φ_2 give from vertex 2, φ_3 give from vertex 3. In triangular co-ordinates the charge density vector is

$$\rho = \xi_1\rho_1 + \xi_2\rho_2 + \xi_3\rho_3 = \alpha\rho$$

Substituting for φ and ρ in the above functional equation, we have

$$L(\varphi) = \sum_{\Delta}\iint_{\Delta}\{\frac{1}{2}\varepsilon[\nabla\alpha\phi]^2 - [\alpha\rho][\alpha\phi]\}dR$$

We can represent the equation in Cartesian coordinate

$$L(\varphi) = \sum_{\Delta}\iint_{\Delta}\{\frac{1}{2}\varepsilon[(\frac{\partial}{\partial x}\alpha\phi)^2 + (\frac{\partial}{\partial y}\alpha\phi)^2] - [\alpha\rho][\alpha\phi]\}dR$$

Let be

$$\frac{\partial}{\partial x}\alpha = b,$$
$$\frac{\partial}{\partial y}\alpha = c$$

$$\iint_{\Delta}[\alpha\rho][\alpha\phi]dR = \iint(\phi^{t}\alpha^{t})(\alpha\phi)dR$$

$$= \phi^{t}\{\iint\alpha^{t}\alpha dR\}\rho$$

$$= \phi^{t} A T^{1,1}\rho$$

Where

$$T^{1,1} = \frac{1}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Then

$$L(\varphi) = \sum_{\Delta}\frac{1}{2}\varepsilon\iint_{\Delta} dR[(b\phi)(b\phi) + (c\phi)(c\phi)] - \sum_{\Delta}\phi^{t}\iint_{\Delta}\alpha^{t}\alpha dR.\rho$$

$$L(\varphi) = \sum\{\frac{1}{2}\varepsilon A[\phi^{t}b^{t}b\phi + \phi^{t}c^{t}c\phi] - A\phi t T^{1,1}\rho\}$$

Generally, we can write

$$L(\varphi) = \sum_{\Delta}\{\frac{1}{2}\phi^{t}P\phi - \phi^{t}q\}$$

The sum of the contribution of all elements is

$$L(\varphi) = \frac{1}{2}\phi^{t}P^{g}\phi - \phi^{t}Q^{g}$$

By differentiating a vector ϕ, we have

$$\frac{\partial}{\partial \phi} L = [P^g + (P^g)^t] - q^g$$

For the minimum of the functional L,

$$\frac{\partial}{\partial \phi} L = P^g \phi - q^g = 0$$

$$P^g \phi = q^g$$

Solving this linear equation for the vector ϕ, we obtain all unknowns φi. Thus we can find potentials of all the discrete points on the domain.