Differential Equations for Entertainment

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INTRODUCTION

Calculus deals with problems of continuous variation. Computations of slopes of curves and areas enclosed by closed curves are simple examples showing the hidden logical power of this branch of mathematics. Calculus was anticipated by Greeks who knew how to use the limiting process to estimate the area of a circle. The modern form of calculus was invented by Sir Issac Newton and Wilhelm Leibniz. Differential equation is an extension of calculus that started to develop rapidly from the beginning of the 18^{th} century. A large number of problems in physics, chemistry, engineering, biology, and social sciences lead to differential equations. Modern technological development would not have been possible withou t the advancement achieved in this subject. A student with a background in elementary calculus could easily grasp the theory and applications of differential equations.

DEFINITIONS

A differential equation is a relationship between derivatives of a function.

Example: If y is a function of x, $\frac{dy}{dx} = 2x$, $\left[\frac{dy}{dx}\right]^2 + 3y = \text{Sin}x$,

$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + b = 0$ are differential equations.

The order of a differential equation is the order of the highest derivative in the equation.

Example: $\frac{d^3y}{dx^3} + \left[\frac{dy}{dx}\right]^2 + 3y = 0$ is a third-order equation.

The degree of a differential equation is the highest power of the highest order derivative in the equation.

Example: $\left[\frac{dy}{dx}\right]^2 + 2xy = 0$ is a second-degree equation.

SOLUTION (INTEGRATION) OF DIFFERENTIAL EQUATIONS

Few types of differential equations can be solved by standard methods. Others need special techniques and, in some cases, only approximate solutions can be found.

SOLUTION BY DIRECT INTEGRATION

Equations of the form $\frac{dy}{dx} = f(x)$ have solutions $\frac{dy}{dx} = \int f(x)dx + A$.

In the same way $\frac{d^2y}{dx^2} = f(x)$ can be solved by two integrations.

Each integration needs inclusion of one arbitrary constant. In general, n^{th}-order differential equation will have n arbitrary constants.

VARIABLE SEPARABLE EQUATIONS

The equation $\frac{dy}{dx} = f(x,y) = \frac{g(x)}{h(y)}$ can be integrated by writing in the form, $h(y)dy = g(x)\,dx$

i.e., $\int h(y)\,dy = \int g(x)dx + A$.

LINEAR DIFFERENTIAL EQUATIONS

Definition: A differential equation is said to be linear, if the linear combination of any two solutions of the equation is also a solution.

Consider the equation $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + bg = 0$
where a and b are constants. If y_1 and y_2 are two solutions of the equation, it is easy to prove that their linear combination, $\alpha y_1 + \beta y_2$, is also a solution.

SOLUTION OF THE LINEAR EQUATION

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (1)$$

Assume that $y = Ae^{mx}$, where m = constant is a solution of (1). Then, on substitution, we obtain the equation

$$am^2 + mb + c = 0 \qquad (2)$$

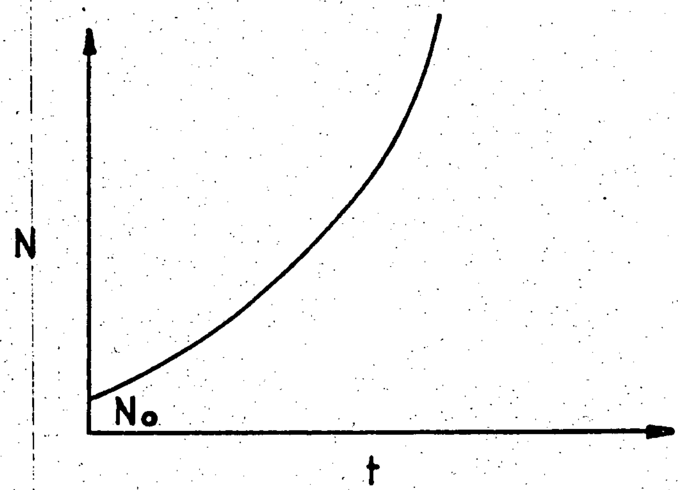
If m_1 and m_2 are the roots of the quadratic (2), A_1e^{m1x} and A_2e^{m2x} are solutions of (2). As the equation is linear, the general solution is $y = A_1 e^{mx} + A_2 e^{mx}$.

POPULATION GROWTH/SPREAD OF A RUMOUR

In the initial stages of growth, the rate of growth dN/dt of population N is proportional to N itself, giving the differential equation

$$\frac{dN}{dt} = kN \qquad (1)$$

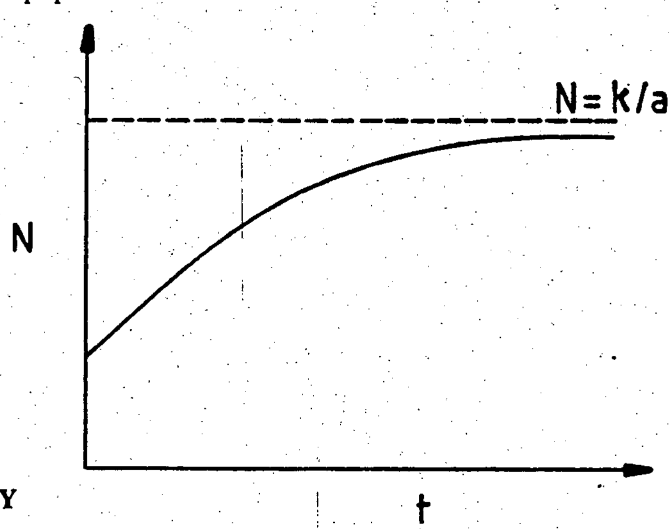
the solution of which is $N = N_0e^{kt}$.



The growth of bacteria, insects, plants, and animals satisfies the above equation approximately. The spread of a rumour in the initial stages will also obey equation (1). As the population increases, the resource limitations tend to decrease the rate of growth. This can be accounted by an equation of the form

$$\frac{dN}{dt} = kN - aN^2. \qquad (2)$$

Equation (2) can be solved by the method of separation of variables. The important point about equation (2) is that an equilibrium population $N = k/a$ exists.



RADIOACTIVE DECAY

The number of nuclei decaying in a radioactive substance is proportional to the total number of undecayed nuclei. Thus we get the differential equation $\frac{dN}{dt} = -kN$, the solution of which is $N = N_0e^{-kt}$.

MOTION OF A RAINDROP

According to Stoke's Law, a spherical object moving in a resistive medium encounters a resistance proportional to its speed. Thus the equation of the motion for a raindrop can be written as

$$m \frac{d^2y}{dt^2} = -k \frac{dy}{dt} + mg \qquad (1)$$

putting, $v = \frac{dy}{dt}$

Equation (1) can be written in the form

$$\frac{dv}{dt} = -(k/m)v + g. \qquad (2)$$

The solution for equation (2) is

$$v = v_0 \ e^{-kt/m} + gm/k \qquad (3)$$

Thus a falling raindrop acquires constant terminal velocity gm/k.

WORM INFECTIONS

Worm infections persist because of the entry of worm eggs into the body. Eggs produced by the worms within the body normally do not accumulate. If N is the number of worms at time t, and c is the rate of entry of eggs, we obtain the differential equation

$$\frac{dN}{dt} = C - kN. \qquad (1)$$

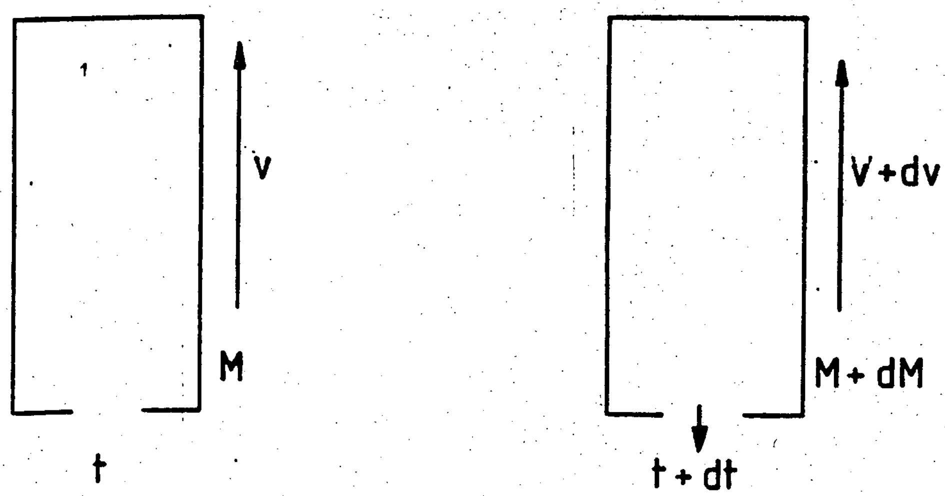
The second term arises because worms inside the body die at a rate proportional to their population. The solution to equation (1) can be written as

$$N = N_0 \ e^{-kt} + C/K. \qquad (2)$$

Results (equation 2), show that protection is possible, if the entry of eggs into the body is prevented.

MOTION OF A ROCKET

A rocket is an object of variable mass. Let M = mass of the rocket at time of ejection, v = velocity at time t, u = velocity of the ejected matter relative to the rocket, and v = rate at which matter is ejected. At time t + dt, the mass and velocity will change to (M + dM) and (v + dv), respectively.



The velocity of the rocket relative to the earth (R,E) = $\uparrow$ v.

The velocity of matter ejected relative to the rocket (M, R) = $\downarrow$u.

$$(M,E) = (M,R) + (R,E)$$

$$= \downarrow u + \uparrow v = (v-u)\uparrow.$$

According to Newton's Law, the change in momentum during a time interval is

dt = force x dt

$$(M + dM)\ (v + dv)\ \text{-}Mv^{+}\ (v\text{-}u)\ rdt = \text{-}Mg\ dt$$

i.e., $vdM + Mdv - (v - u)\ dM = Mgdt$

i.e., $$v\frac{dM}{dt} + M\frac{dv}{dt} - (v\text{-}u)\frac{dM}{dt} = Mg.$$

Since $\frac{dM}{dt} = -r$, the above equation can be solved.

TALE OF THE FOX AND THE RABBIT

Consider an ecosystem consisting of an ample resource of green vegetation, foxes, and rabbits. Let R and F denote the populations of rabbits and foxes at time t. The population of rabbits grows at the expense of the vegetation and is lost when consumed as food by the foxes. The growth of the fox population depends on the availability of rabbits. The rate of removal of rabbits is proportional to the product RF (Law of Mass Action). Thus the rate of equations governing the growth of rabbits and foxes can be written as

$$\frac{dR}{dt} = kR - aRF \qquad (1)$$

$$\frac{dF}{dt} = bRF - hF. \qquad (2)$$

The above equations are nonlinear and, therefore, an exact solution is impossible. However, note that the equilibrium populations R and F correspond to dR/dt = v, dF/dt = v are R = h/b, F = k/a, respectively.

To obtain approximate solutions for equations (1) and (2) about the equilibrium point, we put $R = h/b + r$, $F = k/a + f$, where r and f are small deviations from the equilibrium. When the second-order terms in f and r are neglected we obtain

$$\frac{dr}{dt} = \left[\frac{-ah}{b}\right]f \qquad (3)$$

$$\frac{dt}{dt} = \left[\frac{bk}{a}\right]r. \qquad (4)$$

From equations (3) and (4) we could deduce

$$\frac{d^2r}{dt^2} = -(hk)r \qquad (5)$$

$$\frac{d^2t}{dt^2} = -(hk)f. \qquad (6)$$

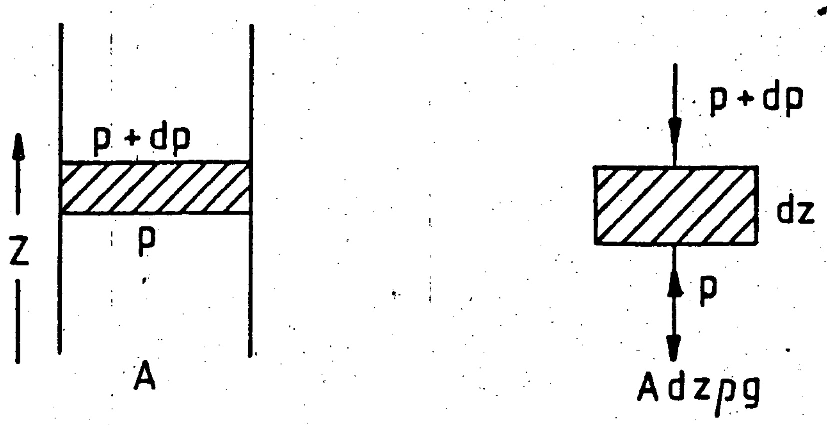
The solutions for equations (5) and (6) are

$$r = A \text{ Cos } (bk)^{1/2}t + b \text{ Sin } (bk)^{1/2}t \qquad (7)$$

$$f = C \text{ Cos } (bk)^{1/2}t + D \text{ Sin } (bk)^{1/2}t. \qquad (8)$$

Thus the populations of foxes and rabbits oscillate about an equilibrium.

VARIATION OF ATMOSPHERIC PRESSURE WITH HEIGHT



Consider a vertical column of air of the cross-sectional area A. Let P = pressure at a height z, P + dP = pressure at a height z + dz, and ρ = density of air at pressure P. Forces acting on the shaded element are indicated below.

For equilibrium

$$(P + dP) A + A\rho gdz = pA$$

i.e., $dP + \rho gdz = 0.$ (1)

Under isothermal conditions, a gas obeys the equation state

$$\rho V = mRT$$

i.e., $P = \rho RT$ (2)

When (2) is substituted for (1), we obtain

$$\frac{dP}{dz} = -(g/KT)P$$

$$P = P_0 \, Exp \, (-g/KT)z.$$